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On the exact solutions of the Lipkin–Meshkov–Glick model

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Abstract

We present the many-particle Hamiltonian model of Lipkin, Meshkov and Glick in the context of deformed polynomial algebras and show that its exact solutions can be easily and naturally obtained within this formalism. The Hamiltonian matrix of each j multiplet can be split into two submatrices associated with two distinct irreps of the deformed algebra. Their invariant subspaces correspond to even and odd numbers of particle–hole excitations.

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1. Introduction

In the 1960s much interest has been devoted to formalisms for treating multiparticle systems and the quality of the approximations involved. To test the validity of the approximations, quasi-exactly solvable models have been proposed (for a definition of a quasi-exactly solvable model see, e.g., [1]). The comparison between the exact solutions and an approximation could give a clear estimate of the quality of the approximation, which could further be applied to more complicated Hamiltonians. Among them of particular interest is the model of Lipkin, Meshkov and Glick (LMG) [2]. Although simple enough to be solved exactly, in some cases the model is not trivial. Few analytic solutions (for numbers of particles up to eight) have been provided by LMG. Numerical solutions were also given for a larger number of particles in the case where the total angular momentum reaches its maximum value. Here we study the exact solutions of the LMG model within the framework of deformed algebras. We first show that the LMG model corresponds to a deformed algebra of polynomial type and then we search for possible solutions associated with the representations of the corresponding deformed algebra. We show that the polynomial algebra introduces a new symmetry in the system, not known previously, and this splits into two submatrices any Hamiltonian matrix to be diagonalized for a specific value of the angular momentum. Moreover, it introduces a

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new quantum number which naturally distinguishes between an even and an odd number of particles.

In the next section we recall the LMG model. In section 3 we briefly introduce deformed algebras and describe how one can use them to find exact solutions of the LMG model for an arbitrary given number of particles and for any specific value j of the angular momentum. In section 4 we presents algebraic and numerical solutions for a system with an odd ($N = 7$) and an even ($N = 8$) number of particles. A general description of some supplementary solutions, inherent to the deformed algebra, is given in section 5. The final section is devoted to a summary.

2. The LMG model

As mentioned in the introduction the model of Lipkin, Meshkov and Glick is a quasi-exactly solvable model developed for treating many-particle systems.

The general method for constructing solvable models is based on the incorporation of some symmetries of the system which give additional integrals of motion and therefore reduce the size of the Hamiltonian matrix to be diagonalized. The Hamiltonian of a many-particle system interacting via a two-body force is a sum of linear and quadratic terms in bilinear products of creation and annihilation operators related to the quantum states of these particles. One starts from the observation that bilinear products of creation and annihilation operators can be considered as elements of a Lie algebra, here related, in particular, to the $SU(2)$ group of the so-called quasi-spin. Lipkin, Meshkov and Glick construct a two N-fold degenerate level model, where N is the number of fermions in the system. The two levels are separated by an energy ϵ. Here we discuss the simplified version of the LMG model where the interaction contains only terms which mix configurations. In the following we use the notation of [5]. Accordingly we introduce fermion operators β_m^+, β_m that create and annihilate holes in the lower level and α_m^+, α_m ($m = 1, 2, \ldots, N$) that create and annihilate particles in the upper level. These operators satisfy the anticommutation relations

$$\{\alpha_m, \alpha_{m'}^+\} = \{\beta_m, \beta_{m'}^+\} = \delta_{mm'} \tag{1}$$

as well as the commutation relations

$$[\alpha_m, \beta_{m'}] = [\alpha_m, \beta_{m'}^+] = [\alpha_m^+, \beta_{m'}] = [\alpha_m^+, \beta_{m'}^+] = 0. \tag{2}$$

The ground state $|0\rangle$ is defined by

$$\alpha_m\,|0\rangle = \beta_m\,|0\rangle = 0. \tag{3}$$

Then the bilinear products

$$j_0 = -\frac{1}{2}\,N + \frac{1}{2}\sum_{m=1}^{N}(\alpha_m^+\,\alpha_m + \beta_m^+\,\beta_m) \tag{4}$$

$$j_+ = \sum_{m=1}^{N}\alpha_m^+\,\beta_m^+ \tag{5}$$

$$j_- = \sum_{m=1}^{N}\alpha_m\,\beta_m \tag{6}$$

form an $su(2)$ algebra. The Hamiltonian under study can be written in terms of these generators as

$$H_{LMG} = \epsilon j_0 + V\,(j_+^2 + j_-^2) \tag{7}$$

where ϵ is the separation energy between the two levels, as introduced above, and V is the interaction strength. For later purposes it is convenient to introduce the strength parameter δ instead of V. They are related by

$$V = \frac{\delta\epsilon}{2N}. \tag{8}$$

The invariant operator of the $su(2)$ algebra

$$j^2 = 1/2(j_+j_- + j_-j_+) + j_0^2 \tag{9}$$

commutes with the Hamiltonian and provides a constant of motion. Thus the Hamiltonian matrix breaks up into submatrices each associated with a different value of j and of the order of $2j+1$. Each state in a j multiplet has a different number of excited particle–hole pairs. The interaction (7) mixes the states within the same j multiplet but cannot mix states with different eigenvalues of j^2. It can only excite or de-excite two particle–hole pairs, or in other words, it can only change the eigenvalue of j_0 by two units. The eigenstates of H_{LMG} therefore have an important property, namely their structure is compatible with Hartree–Fock solutions, so one can study the validity of the random-phase approximation, often used in the treatment of a system of fermions, against the exact solution. From the definitions (1)–(4) it follows that the eigenvalues of j_0 are given by half the difference between the number of particles in the upper level and the number of particles in the lower level. Then the maximum eigenvalue of j_0 and of j is $N/2$. Thus LMG conclude that the largest matrix to be diagonalized is of dimension $2j+1 = N+1$.

Then fixing the number of particles N, LMG diagonalize the largest Hamiltonian matrix associated with $j = N/2$ for several cases. For $N = 2$, 3, 4, 6 and 8 analytical solutions are provided. In addition, the eigenvalues of the multiplet $j = N/2$ are found numerically for $N = 14$, 30 and 50. Here by using the polynomial algebra technique we extend the study of the LMG model to its entire spectrum. We show that for a given number of particles there are two types of states:

(1) states with $j = N/2$, for which the interaction entirely lifts the degeneracy. One of these states corresponds to the lowest eigenvalue. These are the states analysed by LMG and they belong to the largest matrix to be diagonalized of dimension $N+1$;
(2) states with $j < N/2$, for which the eigenvalues are identical to those of a system with $N-2, N-4, \ldots$.

The only difference is that these states are degenerate in a system with N particles but not degenerate in a system with $N-2, N-4, \ldots$ particles, because there they are states of type 1. Thus finding the eigenstates of a system with N particles reduces to the diagonalization of the largest matrix once the states of the system with $N-2$ particles is known.

In the context of the deformed polynomial algebra we show that the largest matrix associated with a given N can be split into two submatrices of dimensions $N/2+1$ and $N/2$ for N even and two submatrices, both of dimensions $(N+1)/2$ for N odd. These submatrices correspond to specific values of the Casimir operator of the deformed algebra. The same statement holds for the largest matrix of a system of $N-2$ particles, and so on. Alternatively, for any j multiplet the corresponding Hamiltonian matrix can be split into two submatrices irrespective of the number of particles. These findings are illustrated in detail for the cases of $N = 7$ and 8 particles.

The splitting of a j multiplet into two submatrices is entirely consistent with the property of the LMG interaction (7) that it can excite or de-excite only two particle–hole pairs. Accordingly, for N even, on the one hand the states with $0, 2, \ldots, N-2, N$ particle–hole pairs form an $N/2+1$ dimension invariant subspace and on the other hand, the states with $1, 3, \ldots, N-1$ particle–hole excitations form another $(N/2)$-dimensional invariant subspace. For N odd the states with $0, 2, \ldots, N-1$ particle–hole excitations and the states with $1, 3, \ldots, N$ particle–hole excitations form two distinct invariant subspaces both of dimension $(N+1)/2$. The deformed polynomial algebra provides a 'quantum number' denoted here by c to distinguish between the eigenvalue of (7) for N even and N odd. For N even one has $c = 0$ and for N odd $c = \pm\frac{1}{4}$.

Moreover, the polynomial algebra technique leads to new representations corresponding to new eigenvalues. Some of these are appropriate to a generalized type of LMG model, some others are meaningless (see section 5).

As far as physics is concerned one should mention that the LMG model possesses states of collective excitations related, for example, in nuclear physics to giant resonances. For this reason Lipkin *et al* studied cases with a number of particles and interaction strength relevant to the treatment of nuclei by the random-phase approximation. In further studies [3,4] they tested the method of linearizing the equations of motion and the diagram summation approximations against exact solutions of their model given in [2]. Recently, a more general Hamiltonian which can test two types of elementary excitations instead of one has been proposed by Lipkin [6].

3. Deformed polynomial algebra

Here we present the LMG model in the context of the deformed algebra. In doing so we follow the general method based on the polynomial deformations of the Lie algebra $sl(2, R)$ as developed in [7].

We start by noting that the Hamiltonian (7) is a particular realization of the more general Hamiltonian

$$H = \epsilon[2J_0 + \delta(J_+ + J_-)] \tag{10}$$

with the parameter ϵ as defined above, δ defined by equation (8) and

$$J_0 = \frac{1}{2}\, j_0 \qquad J_\pm = \frac{1}{2N}\, j_\pm^2. \tag{11}$$

One can show that the operators (11) satisfy the following algebra:

$$[J_0, J_\pm] = \pm J_\pm \tag{12}$$

$$[J_+, J_-] = -\frac{16}{N^2}\, J_0^3 + \frac{2}{N^2}(2j^2 + 2j - 1)J_0 \tag{13}$$

where j is an eigenvalue of (9). These relations define a particular case of a deformed polynomial algebra as studied in [7] with the polynomial in J_0 on the right-hand side of (13) having the coefficients (see the appendix)

$$\alpha = -\frac{16}{N^2} \qquad \beta = 0 \qquad \gamma = \frac{2}{N^2}(2j^2 + 2j - 1) \qquad \Delta = 0. \tag{14}$$

The Casimir operator of this algebra is given by

$$C = J_+ J_- - \frac{4}{N^2} J_0^4 + \frac{8}{N^2} J_0^3 + \frac{2j^2 + 2j - 5}{N^2} J_0^2 - \frac{2j^2 + 2j - 1}{N^2} J_0. \tag{15}$$

The algebra (12) and (13) has two types of representations relevant for our discussion. They are labelled by $q = 1$ and 2, respectively. More precisely, the $q = 1$ representations are defined by the equations

$$\begin{aligned} J_0 \,|JM\rangle &= (M + c)\,|JM\rangle \\ J_+ \,|JM\rangle &= f(M)\,|J, M + 1\rangle \\ J_- \,|JM\rangle &= g(M)\,|J, M - 1\rangle \end{aligned} \tag{16}$$

with $M = -J, \ldots, J$, $J = 0, \frac{1}{2}, \ldots, c \in \Re$ and

$$\begin{aligned} f(M-1)\,g(M) = {} & \frac{1}{N^2}(J - M + 1)(J + M) \\ & \times [2j^2 + 2j - 1 - 4J^2 - 4J - 4M^2 + 4M + 8(1 - 2M)\,c - 24c^2] \end{aligned} \tag{17}$$

where the real number c, constrained by equation (A7) (see the appendix), can take three distinct values given by

$$c = 0 \tag{18}$$

or

$$c = \pm\left[\tfrac{1}{4} j(j+1) - \tfrac{1}{8} - J(J+1)\right]^{1/2}. \tag{19}$$

The $q = 2$ representations are defined in an invariant subspace satisfying

$$\begin{aligned} J_0 \,|J'M'\rangle &= \frac{M'}{2}\,|J'M'\rangle \\ J_+ \,|J'M'\rangle &= f'(M')\,|J', M' + 2\rangle \\ J_- \,|J'M'\rangle &= g'(M')\,|J', M' - 2\rangle \end{aligned} \tag{20}$$

where $J' = 0, 1, 2, \ldots$ and

$$f'(M'-2)g'(M') = \frac{1}{4N^2}(J' - M' + 2)(J' + M')(2j^2 + 2j - 1 - J'^2 - 2J' - M'^2 + 2M') \tag{21}$$

if $M' = -J', \ldots, J' - 2, J'$, and

$$f'(M'-2)g'(M') = \frac{1}{4N^2}(J' - M' + 1)(J' + M' - 1)(2j^2 + 2j - J'^2 - M'^2 + 2M') \tag{22}$$

if $M' = -J' + 1, \ldots, J' - 3, J' - 1$.

The cases $J' = \frac{1}{2}, \frac{3}{2}, \ldots$ are particular in the sense that J' must be equal to j. The relations satisfied by the basis vectors $|jm\rangle$ are

$$\begin{aligned} J_0 \,|jm\rangle &= \frac{m}{2}|jm\rangle \\ J_+ \,|jm\rangle &= f'(m)\,|j, m + 2\rangle \\ J_- \,|jm\rangle &= g'(m)\,|j, m - 2\rangle \end{aligned} \tag{23}$$

with

$$f'(m-2)g'(m) = \frac{1}{4N^2}(j+m)(j+m-1)(j-m+1)(j-m+2). \qquad (24)$$

The Hamiltonian (7) can be associated with the representation $q = 2$ since in this case the invariant subspace is spanned by the vectors $|jm\rangle$ on which the deformed generators act as follows:

$$J_0|jm\rangle = \frac{m}{2}|jm\rangle \qquad (25)$$

$$J_+|jm\rangle = \frac{1}{2N}\sqrt{(j-m-1)(j-m)(j+m+1)(j+m+2)}\,|j, m+2\rangle \qquad (26)$$

$$J_-|jm\rangle = \frac{1}{2N}\sqrt{(j+m-1)(j+m)(j-m+1)(j-m+2)}\,|j, m-2\rangle. \qquad (27)$$

One can see that these relations can be recovered from equations (23) if j is a half-integer and from (20) but with $J' = j$ if j is an integer.

If one now calculates the eigenvalues of the Casimir operator (15) for the representation $q = 2$ and $J' = n$, i.e. an integer, one obtains

$$\langle C\rangle_{J'=n,q=2} = \begin{pmatrix} \frac{1}{2N^2}\,n(n+2)[j(j+1) - \frac{1}{2}(n+1)^2] & M' = n, n-2, \ldots, -n \\ \frac{1}{2N^2}\,(n-1)(n-2)[j(j+1) - \frac{1}{2}n^2] & M' = n-1, n-3, \ldots, -n+1 \end{pmatrix}. \qquad (28)$$

One can therefore see that C has two distinct eigenvalues in the space spanned by $|J'M'\rangle$. This shows that the representation $q = 2$ is reducible. One can easily prove that it can be split into the direct sum

$$(J' = n, q = 2)_{c=0} = \left(J = \frac{n}{2}, q = 1\right)_{c=0} \bigoplus \left(J = \frac{n-1}{2}, q = 1\right)_{c=0} \qquad (29)$$

i.e. the $q = 2$ representation can be decomposed into two $q = 1$ representations and this takes place for $c = 0$ only.

A similar decomposition also holds for half-integer j. In this case one has

$$\left(j = n + \tfrac{1}{2}, q = 2\right)_{c=0} = \left(J = \frac{n}{2}, q = 1\right)_{c=1/4} \bigoplus \left(J = \frac{n}{2}, q = 1\right)_{c=-1/4} \qquad (30)$$

and the eigenvalue of the Casimir operator is the same for $q = 1$ and 2,

$$\langle C\rangle_{j=n+1/2} = \frac{1}{4N^2}\,j(j-1)(j+1)(j+2). \qquad (31)$$

We can then conclude that the Hamiltonian matrix of each j multiplet can be split into two submatrices. Examples are shown in the next section.

From now on we are concerned with $q = 1$ representations only. Then in the invariant subspace defined by equations (16) we obtain the following Hamiltonian matrix to be diagonalized:

$$\langle H\rangle = \begin{pmatrix} 2J+2c & \delta f(J-1) & 0 & 0 & \cdot & \cdot & 0 \\ \delta g(J) & 2J-2+2c & \delta f(J-2) & 0 & \cdot & \cdot & 0 \\ 0 & \delta g(J-1) & 2J-4+2c & \delta f(J-3) & \cdot & \cdot & 0 \\ 0 & 0 & \delta g(J-2) & 2J-6+2c & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2J+2+2c & \delta f(-J) \\ 0 & 0 & 0 & 0 & \cdot & \delta g(-J+1) & -2J+2c \end{pmatrix}. \quad (32)$$

The diagonalization amounts to solving the secular equation

$$\det|\langle H\rangle_{ij} - E\delta_{ij}| = 0. \quad (33)$$

4. The cases $N = 7$ and 8 particles

In this section we give analytic and numerical results for the eigenvalues of (7) obtained by solving the eigenvalue equation (33) for seven and eight particles. For a larger number of particles we checked that we agree perfectly with the numerical values of [2].

4.1. $N = 7$

For $N = 7$ there are $2^7 = 128$ states. The largest Hamiltonian matrix corresponds to $j = \frac{7}{2}$. According to the relation (30) where we have to take $n = (N-1)/2$ this matrix splits into two equal matrices, both having $J = (N-1)/4$. The same procedure applies to the $j-1$ multiplet which is the largest multiplet of $N = 5$ particles, and so on. In table 1 we give the possible j multiplets, their multiplicities m_j and the corresponding values of J. Analytic forms of the eigenvalues can be easily obtained only for $j = \frac{1}{2}$ (trivial case) and $j = \frac{3}{2}$. For $j = \frac{5}{2}$ they are obtained numerically from the secular equation (33) which in this case becomes

$$E^3 - 6cE^2 - \left(\tfrac{13}{4} + \tfrac{4}{7}\delta^2\right)E + \tfrac{15}{2}c + \tfrac{120}{49}c\delta^2 = 0 \quad (34)$$

with $c = \pm\frac{1}{4}$. For $j = \frac{7}{2}$ the secular equation (33) leads to

$$E^4 - 8cE^3 - \left(\tfrac{17}{2} + \tfrac{18}{7}\delta^2\right)E^2 + \left(38 + \tfrac{888}{49}\delta^2\right)cE + \tfrac{105}{16} + \tfrac{75}{14}\delta^2 + \tfrac{135}{343}\delta^4 = 0 \quad (35)$$

with $c = \pm\frac{1}{4}$ which is also solved numerically. The dependence of the eigenvalues on the parameter δ is exhibited in figure 1.

Table 1. Eigenvalues of the Hamiltonian (7) for $N = 7$ particles.

j	m_j	J	Eigenvalues
$\frac{1}{2}$	14	0	$\pm\frac{1}{2}$
$\frac{3}{2}$	14	$\frac{1}{2}$	$\pm\left(\frac{1}{2} \pm \sqrt{1 + \frac{3}{49}\delta^2}\right)$
$\frac{5}{2}$	6	1	
$\frac{7}{2}$	1	$\frac{3}{2}$	

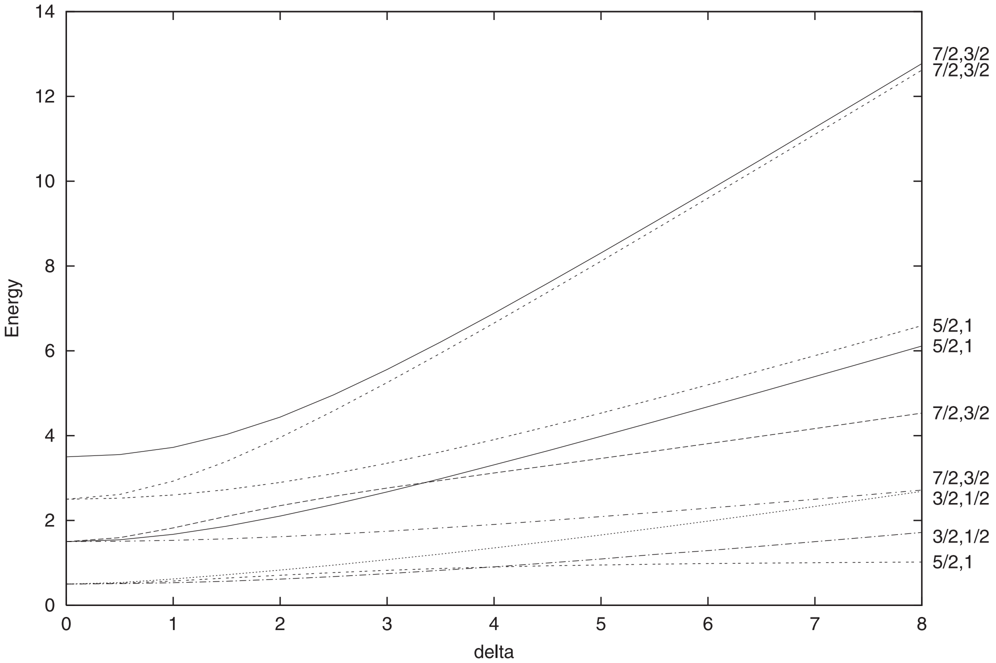

Figure 1. Positive eigenvalues of the LMG Hamiltonian (7), as a function of the parameter δ defined by equation (8) for $N = 7$. The eigenvalues are labelled by j, J and correspond to the rows 3 and 4 of table 1.

4.2. $N = 8$

For $N = 8$ there are 256 states. This case is more fortunate because one can push the analytic calculation further than for $N = 7$. The largest Hamiltonian matrix corresponds to $j = N/2 = 4$. According to (29) where we have to take $n = N/2$ this matrix is split into two submatrices one corresponding to $J = N/4 = 2$ and the other to $J = (N-2)/4 = \frac{3}{2}$. The next to the largest multiplet, with $j = N/2 - 1$, can be seen as the largest multiplet corresponding to $(N-2)$ particles and one can use the decomposition (29) again. In the $(N-2)$-particles case all the eigenvalues are non-degenerate while in the N-particle case the same result is valid for $j = N/2 - 1$, but with some degeneracy for the eigenvalues. In table 2 we exhibit the multiplets j and their multiplicities m_j, the values of J consistent with (29) for each j and the corresponding analytical solutions for the eigenvalues obtained from the secular equation (33). One can check the consistency of these analytic expressions with those of [2][4] by using the relation (8) which gives δ in terms of V/ϵ.

In figure 2 we plot all positive eigenvalues of (7) as a function of the strength δ. One can note an expected degeneracy at $\delta = 0$ and some degeneracy at large values of δ. In particular, the largest eigenvalue with $j = 4$, $J = 2$ becomes degenerate with the largest eigenvalue with $j = 3$, $J = \frac{3}{2}$. The same eigenvalues with opposite sign play the role of the ground state of the system, which thus becomes degenerate for large values of δ.

[4] Note that in [2] above there are some printing mistakes. In the first row of equations (3.5) a factor of 4 is missing in front of the inner square root and in equations (3.5) a factor of 6 is also missing in front of the inner square root.

Table 2. Eigenvalues of the Hamiltonian (7) for $N = 8$ particles.

j	m_j	J	Eigenvalues
0	14	0	0
1	28	0	0
		$\frac{1}{2}$	$\pm\sqrt{1+\frac{1}{64}\delta^2}$
2	20	$\frac{1}{2}$	$\pm\sqrt{1+\frac{9}{64}\delta^2}$
		1	$0\,,\pm\sqrt{4+\frac{3}{16}\delta^2}$
3	7	1	$0\,,\pm\sqrt{4+\frac{15}{16}\delta^2}$
		$\frac{3}{2}$	$\pm\sqrt{5+\frac{33}{64}\delta^2\pm\sqrt{16+\frac{3}{2}\delta^2+\frac{27}{128}\delta^4}}$
4	1	$\frac{3}{2}$	$\pm\sqrt{5+\frac{113}{64}\delta^2\pm\sqrt{16+\frac{19}{2}\delta^2+\frac{275}{128}\delta^4}}$
		2	$0\,,\pm\sqrt{10+\frac{59}{32}\delta^2\pm\sqrt{36-\frac{9}{8}\delta^2+\frac{2025}{1024}\delta^4}}$

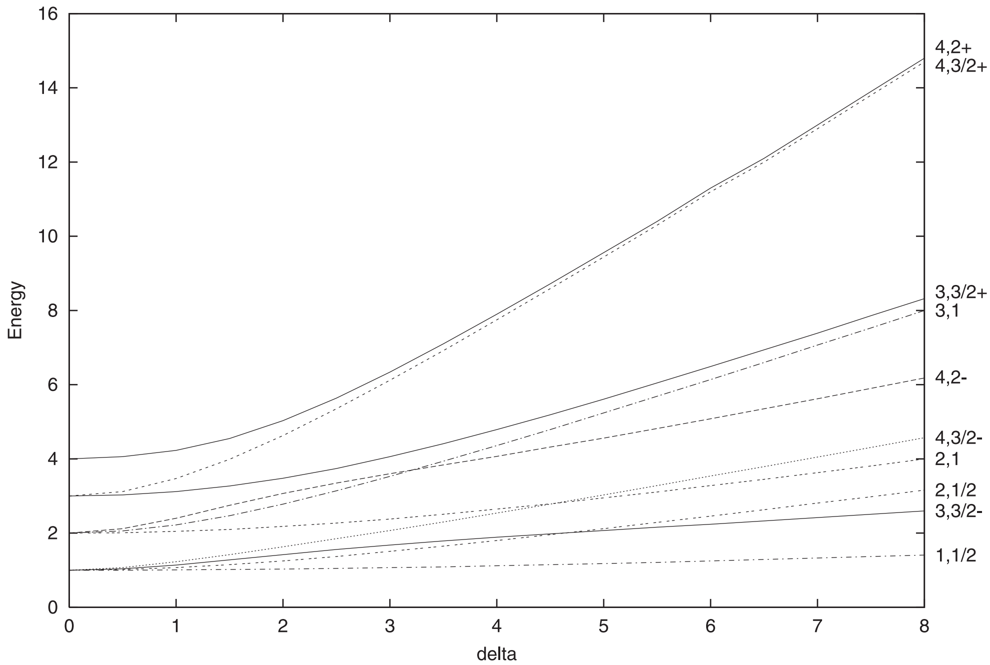

Figure 2. Positive eigenvalues of the LMG Hamiltonian (7), as a function of the parameter δ defined by equation (8) for $N = 8$. The eigenvalues are labelled either by j, J or by j, J, sign when necessary, where sign means the sign in front of the inner square root in the final column of table 2.

5. Supplementary eigenvalues

We have used the $sl(2, R)$ deformed algebra to study the spectrum of the Hamiltonian (7). By construction, this algebra is richer than the $su(2)$ algebra. Its representations have three labels instead of one, as for $su(2)$. Thus the number of representations is larger. Moreover, we can see that once j and c are fixed in (17), one can choose a value of J different from j such that the right-hand side always remains positive. These values have nothing to do with the LMG model. However, it turns out that some eigenvalues of equation (33) associated with these representations are quite similar to those of (7). It would be interesting to find out whether they can be related to a more general Hamiltonian than LMG. This is the subject of a further study.

6. Summary

Here we have presented a calculation of the whole spectrum of the Lipkin–Meshkov–Glick Hamiltonian presented in the context of a deformed polynomial algebra. For any given number of particles N the spectrum first divides into j multiplets of the $su(2)$ algebra. The eigenvalues associated with the largest j are non-degenerate except for $E = 0$. We have shown that the Hamiltonian matrix of each j further splits into two submatrices corresponding to two distinct irreducible representations of the polynomial deformed algebra. These representations bring new 'quantum numbers', one of them allowing us to distinguish between N even and N odd. In order to illustrate the method we have derived explicit analytic expressions for the eigenvalues of the LMG Hamiltonian for $N = 7$ and 8. Our method can be extended to any N.

Furthermore, we have shown that the deformed polynomial algebra related to the LMG model implies a larger spectrum than that of the model itself. Some of the new eigenvalues present characteristics similar to those of the LMG model itself and may lead to a kind of generalized model.

We hope this study could shed new light on the LMG model and could inspire further applications.

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Appendix

For the purpose of self-consistency, here we recall a few results obtained in [7] which are directly exploited in section 3. One of the aims of [7] was to construct finite-dimensional representations of the polynomial deformed algebra

$$[J_0, J_\pm] = \pm J_\pm \tag{A1}$$

$$[J_+, J_-] = \alpha J_0^3 + \beta J_0^2 + \gamma J_0 + \Delta \qquad \alpha, \beta, \gamma, \Delta \in \Re \tag{A2}$$

related to quasi-exactly solvable models. Such representations imply the existence of kets, briefly denoted by $|JM\rangle$, such that

$$\begin{aligned} J_0\,|JM\rangle &= (M/q + c)\,|JM\rangle \\ J_+\,|JM\rangle &= f(M)\,|J, M+q\rangle \\ J_-\,|JM\rangle &= g(M)\,|J, M-q\rangle. \end{aligned} \tag{A3}$$

In fact, the kets are characterized by four labels: the number $J = 0, \frac{1}{2}, 1, \ldots$ related to an eigenvalue of the Casimir operator

$$C = J_+J_- + \frac{\alpha}{4} J_0^4 + \left(\frac{\beta}{3} - \frac{\alpha}{2}\right) J_0^3 + \left(\frac{\alpha}{4} - \frac{\beta}{2} + \frac{\gamma}{2}\right) J_0^2 + \left(\frac{\beta}{6} - \frac{\gamma}{2} + \Delta\right) J_0 \tag{A4}$$

which gives the dimension $2J + 1$ of the representation, the number $M = -J, -J + 1, \ldots, J$ which represents the eigenvalues of J_0, the positive integer q connected to the strength of the raising and lowering operators $J_\pm$ and the shift real number c which also enters the eigenvalues of C. Thus c depends on J.

The highest-weight vectors of each representation impose the constraints

$$\begin{aligned} &f(J) = f(J-1) = \cdots = f(J-q+1) = 0 \\ &g(-J) = g(-J+1) = \cdots = g(-J+q-1) = 0. \end{aligned} \tag{A5}$$

These relations ensure that the dimension of the representation is $2J + 1$ and they lead to the following system of q equations for the number c:

$$\begin{aligned} \alpha \Bigg[c^3 + \frac{3(d-l)}{2q} c^2 + \frac{J^2 - J(d+l) + l^2 - dl + d^2}{q^2} c + \frac{2J - d - l}{2q} c \\ + \frac{2J^2(d-l) - 2J(d^2 - l^2) + d^3 - d^2 l + dl^2 - l^3}{4q^3} + \frac{l^2 - d^2 + 2J(d-l)}{4q^2} \Bigg] \\ + \beta \left[c^2 + \frac{(d-l)c}{q} + \frac{J^2 - J(d+l) + d^2 - dl + l^2}{3q^2} + \frac{2J - d - l}{6q} \right] \\ + \gamma \left(c + \frac{d-l}{2q} \right) + \Delta = 0 \end{aligned} \tag{A6}$$

with $l = 0, 1, \ldots, q-1$. The non-negative integer d, introduced above for convenience, has to take specific values according to J and l. These values have been given in [7]. For the real value of α, β, γ and Δ of equations (14) the system of equations obtained is compatible if and only if $q = 1$ or 2.

For $q = 1$ one has $d = 0$ and one remains with a single equation

$$\alpha c\,[c^2 + J(J+1)] + \beta \left[c^2 + \frac{J(J+1)}{3} \right] + \gamma c + \Delta = 0. \tag{A7}$$

For $q = 2$, one has: (a) $d = 0$ when $l = 0$ and J is an integer or when $l = 1$ and J is a half-integer, (b) $d = 1$ when $l = 0$ and J is a half-integer or when $l = 1$ and J is an integer, with corresponding equations of type (A6).

The representations (A3) are completely specified with

$$\begin{aligned} f(J - kq - q - l) g(J - kq - l) = (k+1) \\ \times \Bigg\{ \alpha \left(\frac{J-l}{q} + c \right)^3 + \beta \left(\frac{J-l}{q} + c \right)^2 + \gamma \left(\frac{J-l}{q} + c \right) + \Delta \\ - \frac{1}{2} \left[3\alpha \left(\frac{J-l}{q} + c \right)^2 + 2\beta \left(\frac{J-l}{q} + c \right) + \gamma \right] k \\ + \frac{1}{6} \left[3\alpha \left(\frac{J-l}{q} + c \right) + \beta \right] k(2k+1) - \frac{\alpha}{4} k^2(k+1) \Bigg\} \end{aligned} \tag{A8}$$

where $l = 0, 1, \ldots, q-1$ and $k = 0, 1, \ldots, (2J - d - l)/q$.

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